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# Large Thermal Deflection of a Cantilever Beam

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**Nonlinear bending of a straight cantilever beam subjected to a temperature distribution that is linear through the thickness and proportional to the cosine of the angle of deformation is analyzed. A closed-form solution is obtained, and numerical results are discussed.**

### Nomenclature

- $A$ = undeformed configuration
- $B$ = deformed configuration
- $h$ = half-depth of beam
- $k$ = coefficient of thermal expansion
- $L$ = length of beam
- $P$ = arbitrary point on deformed middle surface
- $s$ = arc length measured along middle surface
- $T$ = temperature above a fixed datum
- $T_0$ = reference temperature
- $x,y$ = coordinates of point $P$
- $z$ = perpendicular distance from middle surface
- $\epsilon$ = longitudinal fiber strain
- $\theta$ = angle between middle surface normal and $y$ axis

INTEREST in problems concerning theory of the elastica has been widespread among applied mathematicians and engineers from the time of the classical investigations of Bernoulli[1] and Euler[2] to the contemporary studies of Love[3] and others.[4–13] As pointed out by Mitchell,[4] problems involving nonlinear bending of beams are of two basic types; viz., given the free shape and loads, find the deflected shape, or, given the deflected shape and loads, find the free shape. General solutions of the former problem usually entail considerable mathematical difficulty, and only in special cases can a solution be found by other than numerical methods. However, the latter problem, which seldom occurs in practice, possesses solutions that are relatively easy to derive.

The large deflection of a straight horizontal cantilever beam subjected to a vertical point load has been analyzed by Barten[5] and Bisshopp and Drucker.[6] Values of the free-end displacements of a circular-arc cantilever under vertical and horizontal point loads have been given by Conway.[7] Nonlinear bending of a straight horizontal cantilever under uniformly distributed load has been treated by Hummel and Morton,[8] Bickley,[9] Rohde,[10] and Truesdell.[11] Recently, Mitchell[4] presented a unified and more general treatment of several cases considered in Refs. 5–11.

The present note deals with nonlinear bending of a straight cantilever beam subjected to a temperature distribution that is linear through the thickness and proportional to the cosine of the angle of deformation. A closed-form solution is obtained, and numerical results are presented in nondimensional form.

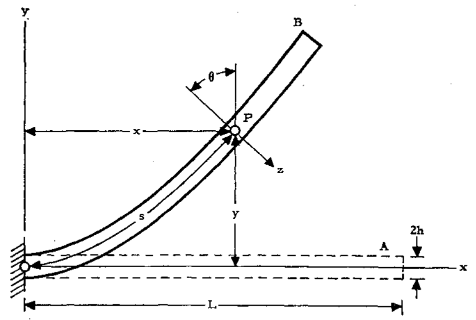


**Fig. 1 Geometry of deformation**

### Basic Equations

Let the cantilever beam shown in Fig. 1 have length $L$ and depth $2h$. Assume that the beam is bent in a principal plane from the undeformed and unstressed configuration $A$ to the deformed configuration $B$ in such a way that the middle surface does not stretch or twist. Cartesian coordinates of an arbitrary point $P$ on the deformed middle surface are denoted by $(x,y)$, $s$ denotes arc length measured along the middle surface, $z$ is the perpendicular distance from the middle surface, and $\theta$ represents the angle between a middle surface normal and the $y$ axis.

Consequently, the engineering strain $\epsilon$ of a longitudinal fiber may be written

$$\epsilon = z(d\theta/ds) \tag{1}$$

Let the beam be exposed to a thermal environment such that its temperature $T$ above some fixed datum is given by the relation

$$T = T_0(z/h)\cos\theta \tag{2}$$

where $T_0$ = const is a reference temperature on the lower surface of the clamped end of the beam. The end $s = L$ is not restrained, and consequently, within the framework of the present theory, the beam will remain stress-free.[14] Thus Hooke's law and Eq. (2) yield

$$\epsilon = kT_0(z/h)\cos\theta \tag{3}$$

where $k$ is the coefficient of thermal expansion.

Necessary differential equations for the determination of $(x,y,\theta)$ follow from geometry of the elastic line (i.e., middle surface) and a comparison of Eqs. (1) and (3), viz.,

$$\begin{aligned} dx/ds &= \cos\theta \\ dy/ds &= \sin\theta \\ d\theta/ds &= (kT_0/h)\cos\theta \end{aligned} \tag{4}$$

### Results

A quadrature of Eqs. (4) subject to the boundary conditions

$$x(0) = y(0) = \theta(0) = 0 \tag{5}$$

gives

$$\begin{aligned} \frac{x}{L} &= \frac{2}{kT_0L/h}\left[\text{arc tan}\left(\exp\frac{kT_0s}{h}\right) - \frac{\pi}{4}\right] \\ \frac{y}{L} &= \frac{1}{kT_0L/h}\ln\left(\cosh\frac{kT_0s}{h}\right) \\ \theta &= \text{arc sin}(\tanh kT_0s/h) \end{aligned} \tag{6}$$

Dimensionless plots of representative elastic curves based on the first two of Eqs. (6) are constructed as shown in Fig. 2.

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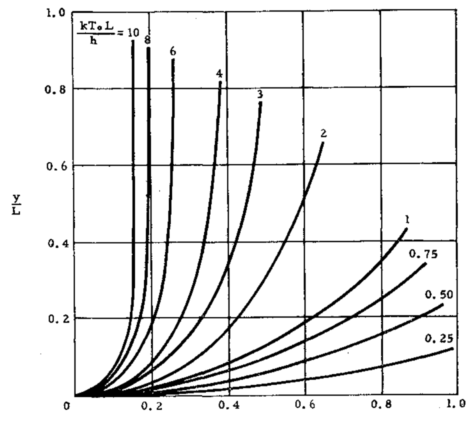


**Fig. 2 Elastic curves**

Values of $y/L$ are plotted vs $x/L$ for several values of $kT_0L/h$. In Fig. 3 the variables $(x/L,\ y/L,\ \theta)$ evaluated at the free end of the beam are plotted vs $kT_0L/h$, and these variables approach the asymptotes $(0, 1, \pi/2)$, respectively.

A vertistat (i.e., an expandable package that may be used for orbital satellite orientation) consists of a tightly rolled thin ribbon of material that forms a long cylindrical shell when expanded.[12] Because of the solar environment, there will be a temperature gradient across the thickness of the tube. The middle surface temperature does not influence the deformation significantly. Consequently, for purposes of analysis, the temperature distribution may be approximated by Eq. (2). In this regard, $2T_0$ is interpreted as the total temperature gradient across the thickness of the tube. As a numerical example, consider the following case:

$$\begin{aligned} k &= 10 \times 10^{-6}\ (^\circ\mathrm{F})^{-1} \\ T_0 &= 10^\circ\mathrm{F} \\ L &= 250\ \mathrm{ft} \\ h &= 0.3\ \mathrm{in.} \end{aligned} \tag{7}$$

In this instance, the elastic curve is the one shown in Fig. 2 for $kT_0L/h = 1$, and from Fig. 3 the coordinates of the free end are approximately

$$(x/L,\ y/L,\ \theta) = (0.865,\ 0.430,\ 0.865) \tag{8}$$

Values of $y/L$ are in good agreement with those obtained by a similar analysis.[12, 13] An expression for $2T_0$ in terms of the incident sun radiation and the tube diameter, thickness, conductivity, and absorptivity is given in Refs. 12 and 13.

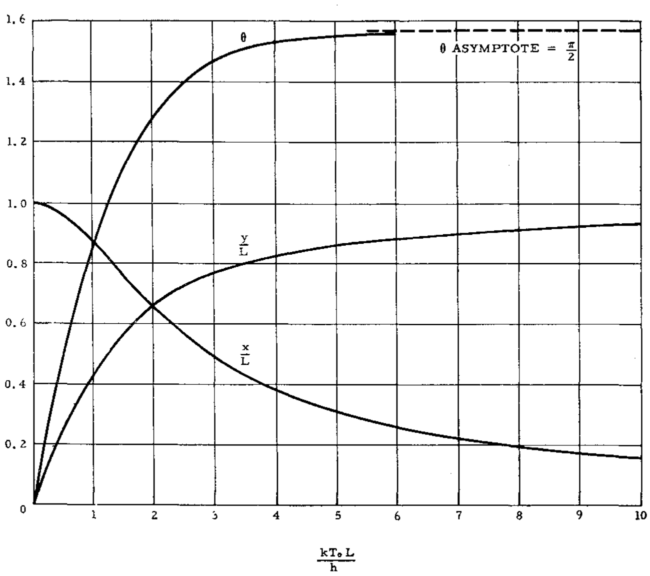


**Fig. 3 Rotation and dimensionless coordinates of free end**

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# Far-Field Approximation for a Nozzle Exhausting into a Vacuum

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Recent trends in technology have led to an interest in the flow field at large distances from a nozzle exhausting into a vacuum. Although the solution to this problem can be obtained by numerical computation using the well known method of characteristics, such solutions frequently are not practicable due to their cost in time or money. Thus an analytic approximation for the density distribution in the far field may be of interest. Such a solution, obtained as part of a more general investigation,[1] is given below.

At distances large compared to the nozzle dimensions, the flow field (as indicated in Fig. 1) approaches radial flow, i.e., the streamlines appear to diverge from a common source point. For a radial flow, the mass flux $\rho u$ varies as $1/x^2$. Since the velocity asymtotically approaches a constant

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